Answers to Coursebook questions – Chapter 2.11

1 The net force on the satellite is $F = G\frac{Mm}{R^2}$ and this plays the role of the centripetal force on the satellite, i.e. $\frac{mv^2}{R}$. Equating the two gives $\frac{mv^2}{R} = G\frac{Mm}{R^2}$, i.e. $v^2 = G\frac{M}{R}$.

For circular motion we have that $v^2 = \left(\frac{2\pi R}{T}\right)^2$ and so $\frac{GM}{R} = \left(\frac{2\pi R}{T}\right)^2$.

Simplifying gives the result.

2 From **Q1** we know that $R^3 = \frac{GMT^2}{4\pi^2}$. The angular velocity is the angle swept per unit time, i.e. $\omega = \frac{2\pi}{T}$, i.e. $T = \frac{2\pi}{\omega}$.

Substituting, $R^3 = \frac{GM(2\pi)^2}{4\pi^2\omega^2} = \frac{GM}{\omega^2}$.

3 From $\frac{mv^2}{R} = G\frac{Mm}{R^2}$ we deduce that $v^2 = G\frac{M}{R}$. Now,
$R = 6400 + 500 = 6900 \text{ km} = 6.9\times10^6 \text{ m}$ and so

$$v = \sqrt{\frac{GM}{R}} = \sqrt{\frac{6.67\times10^{-11}\times6.0\times10^{24}}{6.9\times10^6}} = 7.6\times10^3 \text{ m s}^{-1}.$$

The period is then found from $v = \frac{2\pi R}{T} \Rightarrow T = \frac{2\pi R}{v} = 5.7\times10^3 \text{ s} = 95 \text{ min}$.

4 The period has to be one day, i.e. 24 hours. Then $v = \frac{2\pi R}{T}$ and from $\frac{mv^2}{R} = G\frac{Mm}{R^2}$ we deduce that $v^2 = G\frac{M}{R}$. Combining the results we get

$$R = \sqrt[3]{\frac{GMT^2}{4\pi^2}} = \sqrt[3]{\frac{6.67\times10^{-11}\times6.0\times10^{24}\times(24\times60\times60)^2}{4\pi^2}} = 4.2\times10^7 \text{ m}.$$

5 **a** $$E_P = -\frac{GM_{\text{earth}}M_{\text{moon}}}{r} = -\frac{6.67\times10^{-11}\times5.98\times10^{24}\times7.35\times10^{22}}{3.84\times10^8} = -7.64\times10^{28} \text{ J}.$$

b $$V = -\frac{GM_{\text{earth}}}{r} = -\frac{6.67\times10^{-11}\times5.98\times10^{24}}{3.84\times10^8} = -1.04\times10^6 \text{ J kg}^{-1}.$$

c $$v = \sqrt{\frac{GM_{\text{earth}}}{r}} = \sqrt{\frac{6.67\times10^{-11}\times5.98\times10^{24}}{3.84\times10^8}} = 1.02\times10^3 \text{ m s}^{-1}.$$

6 We must plot the function $E_P = -\frac{GM_{\text{earth}}m}{r} - \frac{GM_{\text{moon}}m}{d-r}$ giving the graph in the answers. Here m is the mass of the spacecraft and d the separation of the earth and the moon (centre-to-centre). Putting numbers in,

$$\begin{aligned}E_P &= -\frac{6.67\times10^{-11}\times5.98\times10^{24}\times3.0\times10^{4}}{r} - \frac{6.67\times10^{-11}\times7.35\times10^{22}\times3.0\times10^{4}}{3.84\times10^{8}-r}\\ &= \frac{1.2\times10^{19}}{r} - \frac{1.5\times10^{17}}{3.84\times10^{8}-r}\\ &= \frac{1.2\times10^{19}/3.84\times10^{8}}{r/3.84\times10^{8}} - \frac{1.5\times10^{17}/3.84\times10^{8}}{1-r/3.84\times10^{8}}\\ &= \frac{3.1\times10^{10}}{x} - \frac{3.9\times10^{8}}{1-x}\end{aligned}$$

where $x = \frac{r}{3.84\times10^{8}}$. In this way the function can be plotted on a calculator to give the graph in the answers (see page 798 in *Physics for the IB Diploma*).

7 **a**

$$\begin{aligned}V &= -\frac{GM}{5R_e}\\ &= -\frac{6.67\times10^{-11}\times5.98\times10^{24}}{5\times6.4\times10^{6}}\\ &= -1.2\times10^{7}\ \text{J kg}^{-1}\end{aligned}$$

b

$$\begin{aligned}E_P &= -\frac{GMm}{5R_e}\\ &= -\frac{6.67\times10^{-11}\times5.98\times10^{24}\times500}{5\times6.4\times10^{6}}\\ &= -6.2\times10^{9}\ \text{J}\end{aligned}$$

8 The net force is the gravitational force and this must point towards the centre of the earth. This happens only for orbit 2.

9 As shown in the text, the reaction force from the spacecraft floor is zero, giving the impression of weightlessness. More simply, both spacecraft and astronaut are in free fall with the same acceleration.

10 The difference is $\Delta U = -\frac{GMm}{R+h} - \left(-\frac{GMm}{R}\right) = \frac{GMm}{R} - \frac{GMm}{R+h} = GMm\left(\frac{R+h-R}{R(R+h)}\right)$.

$\Delta U = GMm\left(\frac{h}{R(R+h)}\right)$. When h is small compared to R this expression is approximately

$$\Delta U = \frac{GMm}{R^2}h = m\frac{GM}{R^2}h = mgh\,.$$

11 The work done by an external agent in moving an object from $r = a$ to $r = b$ at a small constant speed.

12 **a** The potential at the surface of the planet is

$$V = -\frac{GM}{R} = -\frac{6.67\times10^{-11}\times M}{2.0\times10^{5}} = -4.9\times10^{12}\ \text{J kg}^{-1}.$$

Hence $M = 1.5\times10^{28}$ kg.

b The escape speed is obtained from $\frac{1}{2}mv^2 - \frac{GMm}{R} = 0$, i.e. $v = \sqrt{\frac{2GM}{R}}$.

But $V = -\frac{GM}{R}$ and so $GM = -VR$. Substituting, $v = \sqrt{-2V}$.

c $v = \sqrt{-2V} = \sqrt{2\times4.9\times10^{12}} = 3.1\times10^{6}\ \text{m s}^{-1}$

d The work required is $W = m\Delta V$. This is

$W = 1500\times(-1.0\times10^{12} - (-4.9\times10^{12})) = 5.8\times10^{15}$ J.

e We have that $\frac{1}{2}mv^2 + mV_1 = mV_2 \Rightarrow v = \sqrt{2(V_2 - V_1)}$.

This gives $v = \sqrt{2(-2.2\times10^{12} - (-4.9\times10^{12}))} = 2.3\times10^{6}\ \text{m s}^{-1}$.

13 **a** At $r = 0.75$, $g = \frac{GM_P}{(0.75d)^2} - \frac{GM_m}{(0.25d)^2} = 0$. Hence $\frac{M_P}{M_m} = \frac{(0.75d)^2}{(0.25d)^2} = 9$.

b The probe must have enough energy to get to the maximum of the graph. From then on the moon will pull it in. Then

$$W = \frac{1}{2}mv^2 = m\Delta V \Rightarrow v = \sqrt{2\Delta V} = \sqrt{2(-0.2\times10^{12} - (-6.42\times10^{12}))} = 3.5\times10^{6}\ \text{m s}^{-1}.$$

14 We deduced many times that

$$v^2 = \frac{GM}{r}$$

and so

$$E_T = \frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{GMm}{2r} - \frac{GMm}{r} = -\frac{GMm}{2r}.$$

$$E_T = -\frac{6.67\times10^{-11}\times2.0\times10^{30}\times6.0\times10^{24}}{2\times1.5\times10^{11}} = -2.7\times10^{33}\ \text{J}.$$

15 Using $E_K = \frac{GMm}{2r}$, $E_P = -\frac{GMm}{r}$ and $E_T = -\frac{GMm}{2r}$ we deduce that

a B has the larger kinetic energy

b A has the larger potential energy and

c A has the larger total energy.

16 The speed in orbit is given from $\frac{mv^2}{r} = G\frac{Mm}{r^2}$ to be $v^2 = G\frac{M}{r}$.

The kinetic energy is then $E_K = \frac{1}{2}mv^2 = \frac{1}{2}\frac{GMm}{r}$.

The potential energy is $E_P = -\frac{GMm}{r}$

and so the total energy is $E_T = \frac{1}{2}\frac{GMm}{r} - \frac{GMm}{r} = -\frac{GMm}{2r}$.

Since $r = 5R$, $E_T = -\frac{GMm}{10R}$.

17 From **Q16**, $E_T = -\frac{GMm}{2r}$. Since we are told that $E_T = -\frac{GMm}{5R}$ and energy is conserved,

$-\frac{GMm}{2r} = -\frac{GMm}{5R} \Rightarrow r = \frac{5R}{2}$.

18 The space station is in an orbit with orbit radius $2R_e$ and so has speed $v = \sqrt{\frac{GM}{2R_e}}$ with respect to the earth.

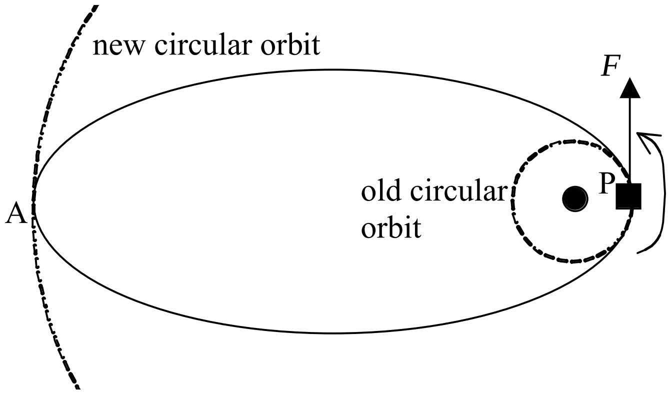
Let the satellite be launched with speed *u* with respect to the space station.
Then the speed with respect to the earth is $u + v$.
Its total energy is therefore $\frac{1}{2}m(u+v)^2 - \frac{GMm}{2R_e}$.
At the escape speed this energy must be zero and so

$u = \sqrt{\frac{GM}{R_e}} - v = \sqrt{\frac{GM}{R_e}} - \sqrt{\frac{GM}{2R_e}} \approx 2.3 \times 10^3 \text{ m s}^{-1}$.

19 **a** The engines do positive work increasing the total energy of the satellite. Since $E_T = -\dfrac{GMm}{2r}$ it follows that the orbit radius will increase.

b Since the kinetic energy is given by $E_K = \dfrac{GMm}{2r}$ and the orbit radius has increased, the speed in the new circular orbit will decrease.

c The firing of the rockets when the satellite is in the lower orbit make the satellite move on an elliptical orbit.
After half a revolution the satellite will be at A and further from the earth than in the original position at P.
As the satellite gets to A its kinetic energy is reduced and the potential energy increases. At A the speed is too low for the new circular orbit and the engines must again be fired to increase the speed to that appropriate to the new orbit. (If the engines are **not** fired at A then the satellite will remain in the elliptical orbit and will return to P.)



20 The tangential component at A is in the direction of velocity and so the planet increases its speed. At B it is opposite to the velocity and so the speed decreases.
The normal component does zero work since the angle between force and displacement is a right angle and $\cos 90^\circ = 0$.

21 The potential energy is given by $E_P = -\dfrac{GMm}{r}$.

This is least when the distance to the sun *r* is the smallest (remember E_P is negative).
Therefore since the total energy is conserved, the kinetic energy and hence the speed are greatest at P.

22 **a** The ratio is

$$\frac{F^B_{\text{moon}}-F^A_{\text{moon}}}{F^B_{\text{sun}}-F^A_{\text{sun}}}=\frac{\dfrac{GM_{\text{moon}}m}{(d_{\text{moon}}-R_e)^2}-\dfrac{GM_{\text{moon}}m}{(d_{\text{moon}}+R_e)^2}}{\dfrac{GM_{\text{sun}}m}{(d_{\text{sun}}-R_e)^2}-\dfrac{GM_{\text{sun}}m}{(d_{\text{sun}}+R_e)^2}}=\frac{\dfrac{M_{\text{moon}}}{(d_{\text{moon}}-R_e)^2}-\dfrac{M_{\text{moon}}}{(d_{\text{moon}}+R_e)^2}}{\dfrac{M_{\text{sun}}}{(d_{\text{sun}}-R_e)^2}-\dfrac{M_{\text{sun}}}{(d_{\text{sun}}+R_e)^2}}$$

.

Numerically this gives

$$\frac{F^B_{\text{moon}}-F^A_{\text{moon}}}{F^B_{\text{sun}}-F^A_{\text{sun}}}=\frac{\dfrac{7.35\times10^{22}}{(3.84\times10^8-6.4\times10^6)^2}-\dfrac{7.35\times10^{22}}{(3.84\times10^8+6.4\times10^6)^2}}{\dfrac{1.99\times10^{30}}{(1.5\times10^{11}-6.4\times10^6)^2}-\dfrac{1.99\times10^{30}}{(1.5\times10^{11}+6.4\times10^6)^2}}\approx 2.2\,.$$

b This indicates the relative importance of the moon.

23 The escape speed is obtained from $\frac{1}{2}mv^2-\frac{GMm}{R}=0$, i.e. $v=\sqrt{\frac{2GM}{R}}$.

But $g=\frac{GM}{R^2}$ and so $GM=gR^2$. Substituting, $v=\sqrt{2gR}$.

24 **a** The net gravitational field strength at the indicated position has magnitude

$$g=\frac{G16m}{(4d/5)^2}-\frac{Gm}{(d/5)^2}=\frac{G25m}{d^2}-\frac{G25m}{d^2}=0\,.$$

b

$$V=-\frac{G16m}{(4d/5)}-\frac{Gm}{(d/5)}=-\frac{G20m}{d}-\frac{G5m}{d}=-\frac{25Gm}{d}\,.$$

25 **a** See **Q1**.

b $T^2=\frac{4\pi^2r^3}{GM}$.

Now $r\approx R$ and $\rho=\frac{M}{\frac{4\pi R^3}{3}}=\frac{3M}{4\pi R^3}$.

Hence, $\frac{M}{R^3}=\frac{4\pi\rho}{3}$.

Substituting, $T=\sqrt{\frac{4\pi^2}{G}\frac{3}{4\pi\rho}}=\sqrt{\frac{3\pi}{G\rho}}$.

c $$\frac{T_{\text{planet}}}{T_{\text{earth}}}=\sqrt{\frac{\rho_{\text{earth}}}{\rho_{\text{planet}}}}\Rightarrow\frac{\rho_{\text{earth}}}{\rho_{\text{planet}}}=\left(\frac{169}{85}\right)^2=3.95\approx4\,.$$

26 **a** We must use the formula $T^2 = \dfrac{4\pi^2 R^3}{GM}$ that we have derived many times already.

Now $g = \dfrac{GM}{R^2} \Rightarrow GM = gR^2$.

Substituting, $T^2 = \dfrac{4\pi^2 R^3}{gR^2} = \dfrac{4\pi^2 R}{g}$.

Hence $T = 2\pi\sqrt{\dfrac{R}{g}}$.

b $T = 2\pi\sqrt{\dfrac{3.4\times10^6}{4.5}} = 5.5\times10^3\ \text{s} = 91\ \text{min}$.

c From $T^2 = \dfrac{4\pi^2 R^3}{GM}$ we deduce that $\dfrac{T_1^2}{T_2^2} = \dfrac{R_1^3}{R_2^3}$.

Hence $\dfrac{91^2}{140^2} = \dfrac{(3.4\times10^6)^3}{R_2^3}$ and so $R_2 = 4.5\times10^6$ m.

The height is therefore $h = 4.5\times10^6 - 3.4\times10^6 = 1.1\times10^6$ m.

27 **a**

$$F = \frac{GM^2}{4R^2}$$

b

$\dfrac{GM^2}{4R^2} = \dfrac{Mv^2}{R^2}$ and so $v^2 = \dfrac{GM}{4R}$.

But $v^2 = \left(\dfrac{2\pi R}{T}\right)^2$ and so $\dfrac{GM}{4R} = \left(\dfrac{2\pi R}{T}\right)^2$.

Hence $T^2 = \dfrac{16\pi^2 R^3}{GM}$.

c $T = \sqrt{\dfrac{16\pi^2(1.0\times10^9)^3}{6.67\times10^{-11}\times1.5\times2.0\times10^{30}}} = 2.8\times10^4\ \text{s} = 7.8\ \text{h}$.

d

$$E_T = \frac{1}{2}Mv^2 + \frac{1}{2}Mv^2 - \frac{GM^2}{2R}.$$

Since

$$v^2 = \frac{GM}{4R}$$

we have that

$$E_T = \frac{1}{2}M\frac{GM}{4R}\times2 - \frac{GM^2}{2R} = \frac{GM^2}{4R} - \frac{GM^2}{2R} = -\frac{GM^2}{4R}.$$

e Since energy is being lost, the total energy will decrease. This implies that the distance R will decrease. (From the period formula in **b** the period will decrease as well.)

f **i** The total energy is $E_T = -\dfrac{GM^2}{4R}$ and the period is $T^2 = \dfrac{16\pi^2 R^3}{GM}$. Combining the two gives $E_T = -\dfrac{GM^2}{4\left(\dfrac{GMT^2}{16\pi^2}\right)^{3/2}}$

or $E_T = -cT^{-3/2}$, where c is a constant. Working as we do with propagation of uncertainties (or using calculus) we have that

$$\frac{\Delta E_T}{E_T} = \frac{3}{2}\frac{\Delta T}{T} \text{ or } \frac{\frac{\Delta E_T}{\Delta t}}{E_T} = \frac{3}{2}\frac{\frac{\Delta T}{\Delta t}}{T}.$$

ii $$\frac{\frac{\Delta E_T}{E_T}}{\Delta t} = \frac{3}{2}\frac{\frac{\Delta T}{T}}{\Delta t} = \frac{3}{2}\times\frac{72\times10^{-6}\text{ s y}^{-1}}{2.8\times10^{4}\text{ s}} = 3.9\times10^{-9}\text{ y}^{-1}$$

g The lifetime is therefore $\dfrac{1}{3.9\times10^{-9}\text{ y}^{-1}} = 2.6\times10^{8}$ y.

28 **a** The pattern is not symmetrical and so the masses must be different. The spherical equipotential surfaces of the right mass are much less distorted and so this is the larger mass.

b The gravitational field lines are normal to the equipotential surfaces.

c From far away it looks like we have a single mass of magnitude equal to the sum of the two individual masses. The equipotential surfaces of a single point mass are spherical.

29 **a** The magnitude of the gravitational field strength is the slope of a potential–distance graph. Drawing a tangent to the curve at $\dfrac{r}{d} = 0.20$ we find a slope of approximately 4.7 N kg^{-1}.

b At $\dfrac{r}{d} = 0.75$ the slope of the graph is zero. But the slope of a potential–distance graph is the magnitude of the gravitational field strength. Hence g is zero at this point.

c Since $g = 0 = \dfrac{GM}{r^2} - \dfrac{Gm}{(d-r)^2} = \dfrac{GM}{(0.75d)^2} - \dfrac{Gm}{(d-0.75d)^2}$ it follows that $\dfrac{M}{m} = \dfrac{(0.75d)^2}{(0.25d)^2} = 9$.

